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IDEMPOTENT METHODS FOR CONTROL AND GAMES

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14. ABSTRACT

Research into application of max-plus (and mode general idempotent) algebra based methods for solution of nonlinear control and estimation problems was undertaken. Such problems are typically solved via the method of dynamic programming, which converts the problems into partial differential equations (PDEs). However that approach is subject to the well-known curse-of-dimensionality when classical grid-based methods are applied to solve the PDEs. The term "curse-of-dimensionality" refers to the fact that the computational complexity grows exponentially fast as the dimension of the state space increases, and has typically forbidden the use of such approaches to real-world applications for well over a half-century. The methods developed in this effort are not subject to the that tremendous computational complexity growth. They are subject to a certain curse-of -complexity; however that is addressed via optimal idempotent projections, which may be instantiated as pruning operations. Specific research topics addressed include extension of these methods to nonlinear stochastic control problems, nonlinear robust estimation problems, quantum-spin control, certain classes of dynamic games, and solution of classes of linear, infinite-dimensional control problems.

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Idempotent Methods for Control and Games

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2 Introduction

Multiple advances were made during the period of the effort. The discussion of such is subdivided by research area below.

3 Max-Plus Methods for Nonlinear Control and Estimation

During this period, efforts on curse-of-dimensionality-free methods based on the max-plus algebra were expanded in several directions. The initial efforts focused on a single class of problems: infinite time-horizon problems where the active control represented an opposing L_2-disturbance process. For reasons of applicability, during this effort, development of the theory necessary to cover wider classes of problems was undertaken. Research in determining the underlying structures which allow phenomenal increases in computational speed for certain classes of problems also continued [16]. Below, we indicate the advances on some aspects of this branch of the research effort.

3.1 Curse of Dimensionality Free Methods for Continuous-Time Stochastic Control

One component of the effort has been in expansion of the applicability of max-plus curse-of-dimensionality-free methods to cover problems in stochastic control. As indicated above, it is now well-known that many classes of deterministic control problems may be solved by max-plus or min-plus (more generally, idempotent) numerical methods. These methods include max-plus basis-expansion approaches [1], [2], [13], [44], as well as the more recently developed curse-of-dimensionality-free methods [44], [43]. It has recently been discovered that idempotent methods are applicable to stochastic control and games. The methods are related to the above curse-of-dimensionality-free methods for deterministic control. In particular, a min-plus based method was found for stochastic control problems [30, 48].

The first such methods for stochastic control were developed only for discrete-time problems. The key tools enabling their development were the idempotent distributive property and the fact that certain solution forms are retained through application of the semigroup operator (i.e., the dynamic programming principle operator). In particular, under certain conditions, pointwise minima of affine and quadratic forms pass through this operator. As the operator contains an expectation component, this requires application of the idempotent distributive property. In the case of finite sums and products, this property looks like our standard-algebra distributive property; in the continuum case, it is familiar to control theorists through notions of strategies, non-anticipative mappings and/or progressively measurable controls. Using this technology, the value function can be propagated backwards with a representation as a pointwise minimum of quadratic or affine forms.

In the recent work, the severe restriction to discrete-time problems was removed. This extension required us to overcome significant technical hurdles. Note that as these methods are related to the max-plus curse-of-dimensionality-free methods of deterministic control, there is a discretization over time, but not over space. One first defines a parameterized set of operators, approximating the dynamic programming operator. Next, one obtains the solutions to the problem of backward propagation by repeated application of the approximating operators. These solutions are parameterized by the time-discretization step size. Using techniques from the theory of viscosity solutions, it is shown that the solutions converge to the viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) associated with the original problem, and consequently to the value function.

The problem is thereby reduced to backward propagation by these approximating operators. The min-plus distributive property is employed. A generalization of this distributive property, applicable to continuum versions is obtained. This allows interchange of expectation over normal random variables (and other random variables with range in $\mathbb{R}^m$) with infimum operators. At each time-step, the solution is represented as an infimum over a set of quadratic forms. Use of the min-plus distributive property allows one to maintain that solution form as one propagates backward in time. Backward propagation is reduced to simple standard-sense linear algebraic operations for the coefficients in the representation. It is also demonstrated that the assumptions on the representation which allow one to propagate backward one step are inherited by the representation at the next step. The difficulty with the approach is an extreme curse-of-complexity, wherein the number of terms in the min-plus expansion grows very rapidly as one propagates. The complexity growth is attenuated via projection onto a lower dimensional min-plus subspace at each time step. At each step, one desires to project onto the optimal subspace relative to the solution approximation. That is, the subspace is not set a priori. In the discrete-time case, it was demonstrated that for some problem classes, this approach might be superior to grid-based methods [30].

The specific class of problems which were addressed are as follows. The dynamics take the form

$$d\xi_s = f^{\mu_s}(\xi_s, u_s)\,ds + \sigma^{\mu_s}(\xi_s, u_s)\,dB_s, \quad \xi_t = x \in I\!\!R^n, \tag{1}$$

where $f^m(x,u)$ is measurable. The u_s and μ_s are control inputs taking values in $U \subset I\!\!R^p$ and $\mathcal{M} \doteq]1, M[\doteq \{1, 2, \cdots, M\}$, respectively. In practice, it is useful to allow both a continuum-valued control component and a finite set-valued component, where the latter is used to allow approximation of more general nonlinear Hamiltonians, c.f. [44],[40] for motivation. Also, $\{B_\cdot, \mathcal{F}_\cdot\}$ is an l-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}_0$ contains all the P-negligible elements of $\mathcal{F}$ and $\sigma^m(x,u)$ is an $n \times l$ matrix-valued diffusion coefficient.

The payoff (to be minimized) is

$$J(t, x, u_\cdot, \mu_\cdot) \doteq \mathbf{E}\left\{\int_t^T l^{\mu_s}(\xi_s, u_s)\,ds + \Psi(\xi_T)\right\}$$

where

$$\Psi(x) \doteq \inf_{z_T \in Z'_T} \{g_T(x, z_T)\},$$

where $l^m(x,u)$ and $g_T(x, z_T)$ are measurable, and $(Z'_T, d_{Z'_T})$ is a separable metric space. The value function is

$$V(t,x) = \inf_{u_\cdot \in \mathcal{U}_t, \mu_\cdot \in \widetilde{\mathcal{M}}_t} J(t, x, u_\cdot, \mu_\cdot), \tag{2}$$

where $\mathcal{U}_t$ (*resp.* $\widetilde{\mathcal{M}}_t$) is the set of $\mathcal{F}_t$-progressively measurable controls, taking values in U (*resp.* $\mathcal{M}$) such that there exists a strong solution to (1). Further assumptions, and the full theory and algorithm, may be found in the references [24, 27, 31, 34].

3.2 Max-plus Methods in Estimation

The max-plus curse-of-dimensionality-free approach is also being extended to problems in estimation, specifically for nonlinear estimation problems. Robust approaches to estimation were explored heavily in the late 1990s, c.f., [14, 50, 18, 52, 17]. Some initial efforts on application of max-plus approaches to such appeared in [13, 51, 51]. With the development of curse-of-dimensionality-free algorithms, it was decided to port over this new technology to the domain of robust nonlinear filtering.

One class of problems that have been addressed is that of robust estimation for problems in attitude estimation on SO(3), where SO(3) denotes the special orthogonal group, which is a particularly useful means of describing rotations of objects in $I\!R^3$ and the dynamics thereof. As linear-dynamics models are not appropriate, max-plus approaches may be particularly relevant. As initial investigation into this domain appears in [57].

Anotehr area of investigation was in application of curse-of-dimensionality-free methods to nonlinear robust estimation, particularly in tandem with the set-valued estimation techniques of Petersen et al. This is documented in [19].

3.3 Control of higher dimensional quantum systems

The curse-of-dimensionality-free approach was developed and tested on problems in the control of quantum spin, where the effort was concentrated on the two-qubit problem, where the relevant state-space is special unitary group, SU(4). This research is documented on [56, 59, 60].

We briefly indicate the basic problem. In general, the quantum systems that were considered, evolve on the Lie group SU(2^n) denoted also by $\mathbf{G}$. The system dynamics are given by

$$\frac{dU}{dt} = \left\{ \sum_{k=1}^{M_0} v_k(t) H_k \right\} U, \qquad U \in \mathbf{G} \tag{3}$$

with initial condition, say $U(r) = U_0$, and control, $v \in \mathcal{V}_g \subset L_\infty([0,\infty); \mathbb{R}^{M_0})$. In (3) $H_1,\, H_2\, \ldots\, H_{M_0}$ constitute a set of right invariant vector fields, which correspond to the set of available one and two qubit Hamiltonians. The span of the set $\{H_1,\, H_2\, \ldots\, H_{M_0}\}$ plus all possible Lie bracket operations thereof, is assumed to be the Lie algebra $\mathfrak{g}$ of the group $\mathbf{G}$. It follows from these assumptions that the time to move the state from the identity element to any other point on the group, is bounded. Given a control signal $v \in \mathcal{V}_g$ and an initial unitary $U(r) = U_0$, the solution to (3) at time t will be denoted by $U(t; v, r, U_0)$.

The class of problems of interest involves determining the optimal trajectory to move between any two points in $\mathbf{G}$ while minimizing an associated cost function, $\tilde{C}_0(\cdot,\cdot)$. The first-passage time function between any two points is denoted by $\hat{t}_{U_1,U_2}(v) := \inf\{t > 0 : U(0) = U_1, U(t) = U_2\}$. The optimal control problem of evolving the state from U_1 to U_2 has a value function given by

$$
\begin{aligned}
&\tilde{C}_0(U_1, U_2) \doteq \inf_{v \in \mathcal{V}_g} \left\{ \int_0^{\hat{t}_{U_1,U_2}(v)} \sqrt{v(s)^T R v(s)}\, ds \right\} \\
&\mathcal{V}_g \doteq \left\{ v \in L_\infty([0,\infty); \mathbb{R}^{M_0}) \,\middle|\, |v(t)| = 1\, \forall t \in [0,\infty) \right\}.
\end{aligned}
$$

The (symmetric, positive-definite) weight matrix, R, reflects the relative difficulty of generating each element of the control vector. If R is diagonal then it can be interpreted as a weight matrix where the diagonal terms are the costs associated with the different control directions available.

One may approach the problem via the min-plus curse-of-dimensionality-free framework. However as previous theory (c.f., [44]) did not consider optimal cost functions with stopping time constraints, a relaxation was formulated. The relaxation was generated by introducing a fixed terminal time, T, and a terminal penalty cost to yield

$$
\begin{aligned}
&J_s^\epsilon(U_0, v) = \left\{ \int_s^T \sqrt{v(t)^T R v(t)}\, dt + \frac{1}{\epsilon}\varphi(U(T; v, s, U_0)) \right\} \\
&C_s^\epsilon(U_0) = \inf_{v \in \mathcal{V}_g^e} J_s^\epsilon(U_0, v),
\end{aligned}
\tag{4}
$$

where $\varphi(\cdot)$ is a continuous real valued non-negative function that is zero only at the identity element. Obviously, the function penalizes terminal states away from the identity. The control set $\mathcal{V}_g^e$ for this relaxed formulation is defined as $\mathcal{V}_g^e \doteq \left\{ v \in L_\infty(0,\infty); I\!R^{M_0}) \,\middle|\, \|v(t)\| \in \{0,1\}\ \forall t \in [0,\infty) \right\}$. This extended control set, $\mathcal{V}_g^e$, ensures that once the target set is reached, the cost $J_s^\epsilon(\cdot)$ in (4) does not continue accruing. This extension of the control ensures that the relaxed cost function converges to the original cost function [56]. The terminal cost was taken as

$$
\varphi(U) = \mathrm{tr}[(I-U)(I-U)^\dagger] = \mathrm{tr}[2I - U - U^\dagger] = 2\mathrm{tr}[I] - 2\, Re\{\mathrm{tr}[U]\}.
$$

Convergence rates and error estimates are given in [56], while results also appear in [59, 60].

4 Payoff Suboptimality Induced by Approximation of the Hamiltonian

Max-plus curse-of-dimensionality-free methods for deterministic control problems are developed using an HJB PDE model corresponding to a control problem where one controller, the actual controller, takes values in $\mathbb{R}^n$, while a second is a switching controller, which transitions the system between various quadratic models (with affine terms as well). The second controller may be used a a means for approximating a general nonlinear Hamiltonian as a pointwise maximum (or a pointwise minimum) of quadratic forms. Note that any semiconvex [semiconcave] Hamiltonian may be expanded as a max-plus [min-plus] linear combination of quadratic forms. This leads to the questions regarding the difference between the solutions of two HJB PDEs, where one is given as a maximum of quadratic forms. A more difficult question, and one which has not previously been significantly addressed, regards estimation of the difference between the payoff one would achieve from a controller based on the original HJB PDE, and the payoff one would achieve with a controller based on an approximating HJB PDE. This second question is quite a bit more delicate that the first. Both are addressed in [23].

Specifically, the originating HJB PDE problem is given by:

$$0 = -H(x, \nabla V) = -\sup_{w \in \mathbb{R}^k} \left[f'(x,w) \nabla V + l(x,w) \right], \quad V(0) = 0$$

where $x \in \mathbb{R}^n$. More specifically, one seeks the particular viscosity solution which is the value function of the optimal control problem with dynamics and running cost

$$\begin{aligned} \dot{\xi}_t &= f(\xi_t, w_t) \doteq g(\xi_t) + \sigma(\xi_t) w_t, \qquad \xi_0 = x, \\ l(\xi_t, w_t) &\doteq L(\xi_t) - \frac{\gamma^2}{2} |w_t|^2. \end{aligned}$$

That is, the value function is

$$\widehat{V}(x) = \sup_{w \in L_2((0,\infty); \mathbb{R}^k)} \sup_{T < \infty} \int_0^T l(\xi_t, w_t)\, dt.$$

The approximating HJB PDE is

$$0 = -\widetilde{H}(x, \nabla V) = -\max_{m \in \mathcal{M}} \{ H^m(x, \nabla V) \}, \quad V(0) = 0$$

where $\mathcal{M} = \{1, 2 \ldots M\}$, and the H^m take the form

$$H^m(x,p) = \tfrac{1}{2}x' D^m x + \tfrac{1}{2\gamma^2} p' \sigma^m (\sigma^m)' p + (A^m x)' p + (l_1^m)' x + (l_2^m)' p + \alpha^m.$$

The associated approximating control problem is given by

$$\widetilde{V}(x) \doteq \sup_{T<\infty} \sup_{\mu \in \mathcal{D}_\infty} \sup_{w \in \mathcal{W}} \int_0^T L^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|w_t|^2 \, dt,$$

where

$$\begin{aligned} L^m(x) &= \tfrac{1}{2}x' D^m x + (l_1^m)' x + \alpha^m, \\ \dot{\xi} &= A^{\mu_t}\xi_t + l_2^{\mu_t} + \sigma^{\mu_t} w_t, \qquad \xi_0 = x, \end{aligned}$$

and $\mathcal{D}_\infty = \{\mu : [0,\infty) \to \mathcal{M} \mid \text{ measurable } \}$.

It is assumed that H and $\widetilde{H}$ are *close* in the sense that there exists $\theta > 0$ such that, for all $x, p \in I\!R^n$ such that $\widetilde{H}(x,p) \le 0$, one has

$$\widetilde{H}(x,p) \le H(x,p) \le \widetilde{H}(x,p) + \theta \left[|x|^2 + |p|^2 \right].$$

Note that the coefficient θ parameterizes the degree of closeness between H and $\widetilde{H}$. As we are dealing with max-plus vector spaces, $\widetilde{H}$ approximates H from below (c.f. [44]), and so this approximation assumption is one-sided.

One finds that the controller based on solution of the approximating HJB PDE satisfies

$$\lim_{T\to\infty} \int_0^T l(\xi_t, w_t)\, dt \ge \widehat{V}(x) - \theta(1 + K_g^2) C_2 |x|^2,$$

where more details may be found in [23].

5 Max-plus methods for solution of infinite-dimensional Riccati equations

The objective of this portion of the effort is to generalize the approach of [29] to classes of infinite dimensional integro-differential Riccati equations [7], leading to a new fundamental solution for these classes of equations.

As in the finite dimensional case [29], this fundamental solution is based on the max-plus dual of the dynamic programming evolution operator (or semigroup) of an associated control problem. Here, the fundamental solution developed is for a specific class of infinite-dimensional integro-differential Riccati equations that was originally motivated by a related problem concerning the amplification of optical signals in optical networks [8]. However, the principle can easily be extended to other infinite dimensional Riccati equations, such as those related to diffusion PDEs. See [10]. For the sake of brevity, only the first class of problems is discussed here.

The specific class of Riccati equations considered takes the form of an operator equation, given by

$$\dot{\mathcal{P}}_t = \mathcal{P}_t\, A + A'\, \mathcal{P}_t - \nabla' \mathcal{P}_t - (\nabla' \mathcal{P}_t)' + \mathcal{P}_t\, \sigma\, \sigma'\, \mathcal{P}_t + \mathcal{C}\,, \tag{5}$$

where $\mathcal{P}_t$ is a time-indexed self-adjoint integral operator, with the specifics of the remaining notation given in [9]. In terms of the operator kernel P_t corresponding to integral operator $\mathcal{P}_t$, an equivalent form of this equation (with details of the notation also in [9]) is

$$\dot{P}_t = P_t\, A + A'\, P_t + \partial_1 P_t + \partial_2 P_t + (P_t\, \sigma) \circledast (\sigma'\, P_t) + C. \tag{6}$$

The theory yielding the aforementioned fundamental solution proceeds by considering an associated infinite-dimensional optimal control problem defined on a finite time horizon. This control problem is constructed such that the associated value function exhibits quadratic growth with respect to the state variable, where this growth is determined by the solution of the Riccati equation in question. Specifically, the integral operator (or kernel thereof) that generates this quadratic growth is the solution of the appropriate Riccati equation. Propagation of the max-plus dual of the solution of this associated control problem is used to characterize the fundamental solution.

In posing the optimal control problem, an appropriately generalized $\mathcal{L}_2$-dissipative running cost is utilized, along with a quadratic terminal cost. This cost is defined with respect to trajectories generated by a infinite dimensional linear system. The optimal control problem enjoys an explicit solution, with the value obtained being a quadratic functional. This value may be propagated in time by the application of a dynamic programming evolution operator $\mathcal{S}_t$, which is max-plus linear on a space of semiconvex functionals. By taking the max-plus dual of this evolution operator, an analogous evolution operator $\mathcal{B}_t^{\oplus}$ may be defined in the dual space. That is, $\mathcal{B}_t^{\oplus}$ propagates the dual of the value of the optimal control problem, and hence the solution of the Riccati equation. Critically, this can be achieved for quite general initial conditions of the Riccati equation, thereby prescribing the fundamental solution via $\mathcal{B}_t^\oplus$. Furthermore, as $\mathcal{B}_t^\oplus$ is a max-plus integral operator, this dual space propagation may be described in terms of the propagation of the kernel B_t of $\mathcal{B}_t^\oplus$. As this propagation follows by a max-plus convolution of quadratic functionals, an explicit computation for propagating B_t is rendered possible. A recipe summarizing the overall approach is provided in [11] (see also [9]).

The following example exhibits a significant reduction in computation time for the solution of the Riccati equation, when compared with standard integration techniques. A scalar-valued Riccati equation of the form (6) is considered, in which $A = -2$, $\sigma = 1/\sqrt{2}$, $C = 1/3$. The spatial interval Λ is defined by $L = 2$. The Riccati equation (6) is solved numerically via a Runge-Kutta method (RK45) and via the dual-space propagation method. Approximation errors were computed with respect to a fine grid RK45 computation. These errors are illustrated in Figure 1 (scaled via the natural logarithm), where a considerable computational advantage of the dual-space propagation is demonstrated.

6 Min-max spaces and complexity reduction in min-max expansions

In max-plus curse-of-dimensionality-free methods for solution of deterministic nonlinear control problems, one uses the fact that the value function lies in the space of semiconvex functions (in the case of maximizing controllers), and approximates this value using a truncated max-plus basis expansion. In some classes, the value function is actually convex, and then one specifically approximates with suprema (i.e., max-plus sums) of affine functions. Note that the space of convex functions is a max-plus linear space, or moduloid. In extending those concepts to game problems, one finds a different function space, and different algebra, to be appropriate. Specifically, one considers functions which may be represented using infima (i.e., min-max sums) of max-plus affine functions. It is natural to refer to the class of functions so represented as the min-max linear space (or moduloid) of max-plus hypoconvex functions. In this component of the effort, the space of max-plus

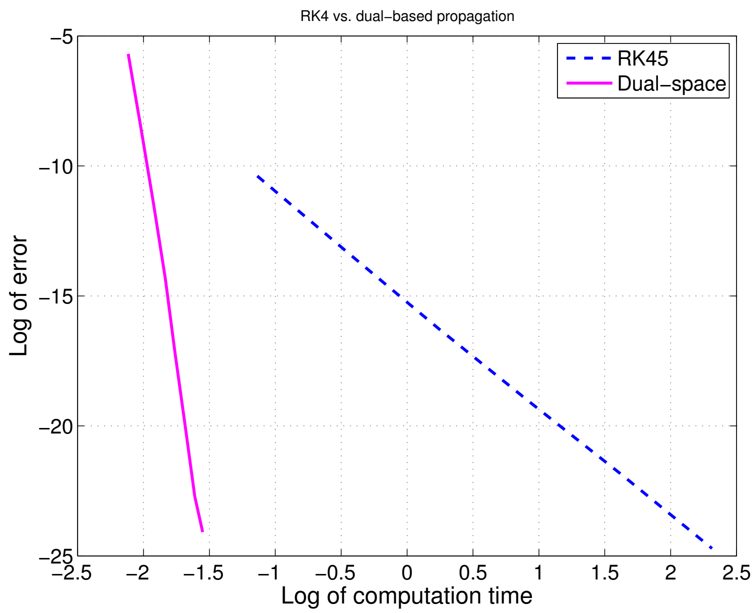

Figure 1: Approximation error versus computation time for standard (RK45) and dual-propagation methods.

hypo-convex functions was examined, as well as the associated notion of duality and min-max basis expansions. With this, one can extend the curse-of-dimensionality-free methods for solution of control problems, to solution of zero-sum dynamic games. The critical step in such development is that of finding reduced-complexity expansions which approximate a function as well as possible. A solution to this complexity-reduction problem in the case of min-max expansions was obtained. Some more detail is now given.

Certain function spaces (to be described below) may be spanned by infima of max-plus affine functions, that is, any element of the space may be represented as an infimum of a set of max-plus affine functions. In the min-max algebra (more properly, the min-max commutative semiring), the addition and multiplication operations are defined as

$$a \oplus^{\vee} b \doteq \min\{a, b\}, \qquad a \otimes^{\vee} b \doteq \max\{a, b\},$$

operating on $\overline{I\!R} \doteq I\!R \cup \{-\infty\} \cup \{+\infty\}$, where one may note $+\infty \oplus^{\vee} b = b$ for all $b \in \overline{I\!R}$ and $-\infty \otimes^{\vee} b = b$ for all $b \in \overline{I\!R}$. As usual, the max-plus algebra (more properly, the max-plus commutative semifield) is defined by

$$a \oplus b \doteq \max\{a, b\}, \qquad a \otimes b \doteq a + b,$$

operating on $I\!\!R^- \doteq I\!\!R \cup \{-\infty\}$.

By definition, any function in a space spanned by infima of of max-plus affine functions has an expansion, $f(x) = \inf_{\lambda\in\Lambda} \psi_\lambda(x)$, for some index set Λ, where the ψ_λ are max-plus affine. If the expansion is guaranteed to be countably infinite, one would write

$$f(x) = \inf_{i\in\mathbf{N}} \psi_i(x) = \bigoplus_{i\in\mathbf{N}}^{\vee} \psi_i(x) \doteq \bigoplus_{i\in\mathbf{N}}^{\vee} a_i \otimes \phi_i(x),$$

where the ϕ_i are max-plus linear. One may refer to this as a min-max basis expansion, and think of the required set of such ϕ_i as a min-max basis for the space. The max-plus analog of this concept consists of max-plus vector spaces (more typically referred to as moduloids [3] or as idempotent semimodules [5], [22]) and max-plus basis expansions. These are heavily employed in max-plus methods for solution of HJB PDEs (c.f., [1], [2], [13], [44], [42]).

In the solution of stochastic control problems via the max-plus (or min-plus) curse-of-dimensionality-free method [37], [30], [35], [47] (also, see Section 3.1 above), an important question is how to represent a function as closely as possible with a fixed number of such basis functions (a truncated max-plus expansion). More specifically, given a function represented as a max-plus sum (i.e., a pointwise maximum) of M (standard-algebra) affine functions, one wants to find the set of N (with $N < M$) affine functions whose max-plus sum best approximates the original max-plus sum from below.

Consider a similar problem, but geared toward a game application (i.e., a max-plus stochastic control application [12]). See [36] for more detail on the originating game problem application. In this case, given a set of M max-plus affine functions, one would like to approximate their min-max sum (i.e., pointwise minimum) from above with a set of N ($N < M$) max-plus affine functions. It can be seen that this problem has the same abstract form as the problem considered in [35], but here the standard and max-plus algebras are replaced by the max-plus and min-max algebras, respectively. The problem reduces to minimization of a max-plus hypo-convex monotonic function over a max-plus cornice, where a cornice is a set formed from the upward (or downward, as appropriate) cones of the points in the convex hull of a set of generating points and a function will be max-plus hypo-convex if its hypograph is convex. For this problem class, one finds that the optimal approximating set of N functions, consists of elements taken from the original set of M functions. In other words, pruning is optimal, and represents the optimal min-max projection onto a min-max space of dimension N.

Some additional remarks on this topic deserve mention. First, the space of max-plus hypo-convex functions, also known as the space of sub-topical functions [55], is the space of monotonically increasing (with respect to the standard partial order on the domain) functions with Lipschitz constant one. (There are obvious generalizations to any fixed Lipschitz constant, etc.) Second, it was shown that the evaluation of any pruning option was remarkably simple, requiring evaluation of the constituent max-plus affine functionals only at the associated "crux points". For more information, one may see [15, 28, 36]

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